# A NOTE ON SPECIAL SOLUTIONS OF THE HYDRODYNAMICAL EQUATIONS

V. A. Bubnov

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The author discusses the possibility of adjusting the Navier–Stokes equations to fit discontinuous flows.

Present-day hydrodynamics is based on the Euler equations for flows of an ideal fluid and the Navier–Stokes equations, in which a correction is introduced for the forces of internal friction. It is well known that in the description of irrotational flows the solutions of the Euler equations concur with the experimental data within certain limits. In the case of rotational flows, on the other hand, the solutions lead to the well-known theorem of Helmholtz on the invariance of the vortex lines. The ultimate implication of the latter statement is that the Euler equations fail to account for the formation and disappearance of vortices.

If we postulate that the main reason for this situation is the presence of internal friction in the fluid, the equations for a viscous fluid in the Navier–Stokes form should explain the vortex-formation process. But the existing solutions of these equations merely explain the vortex-decay process, i.e., their annihilation. Observation of vortices in the atmosphere, however, disclose as well the possibility of vortices in the form of whirlwinds or tornadoes, which are not annihilated upon meeting an obstacle in their path.

All of the foregoing suggests that the hydrodynamical equations in the Navier–Stokes form do not incorporate physical conditions characterizing the inception and annihilation of vortices. It is clearly necessary to inject some meaningful adjustments into these equations.

The first investigations in this direction were carried out by Kasterin [1] within the scope of an ideal fluid. He assumed that the Euler equations are merely a first approximation for the description of vortex flow fields. The second approximation should allow for the discrete structure of the gas and the continuous variation of the fundamental hydrodynamical variables. For example, it must be assumed within the ideal fluid context that the hydrodynamic velocity suffers a discontinuity at the boundary between potential and vortex flows. Adopting this notion of a discontinuous velocity change as his foundation, Kasterin derived equations describing the vortex field in an ideal fluid.

The molecular–kinetic foundation of this notion was set down by Predvoditelev [2], who relied on Maxwell's method, which is completely divorced from the solution of the Boltzmann equation and permits the transition to the hydrodynamical equations for any distribution function. In order to deduce the equations of motion of a viscous fluid in the Navier–Stokes form Maxwell had to assume equal approach velocities of two colliding molecules. This means that his continuum has a filamentary structure in motion, i.e., the minimum streamer dimensions correspond to the separation of the colliding molecules.

This weakness of Maxwell's hypothesis was brought to attention by Predvoditelev [2]. In place of Maxwell's hypothesis he postulated new conditions regarding the approach velocities of two colliding molecules, making his ultimate transition to a continuum under these conditions. A test of this theory in the phenomenon of acoustic dispersion yielded good agreement with experiment over a wide range of Knudsen numbers [3].

Kasterin's notion is amenable to a relatively simple interpretation by finite-difference methods.

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Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.

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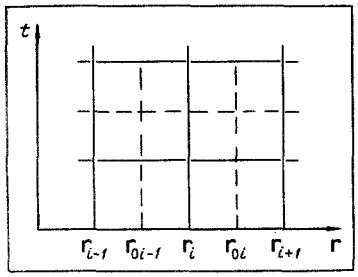


Fig. 1. The net-point unit cell.

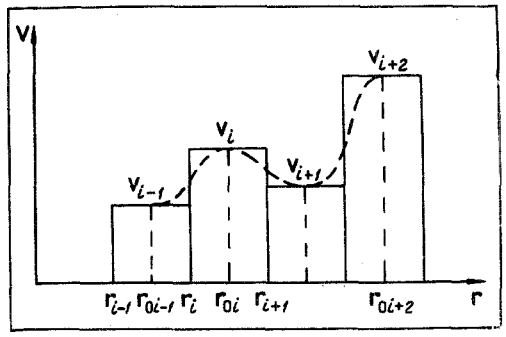


Fig. 2. Discontinuous velocity variation.

In the space occupied by the fluid we construct the net-point unit cell (Fig. 1) so that its side is determined by the quantity $\mathbf{r}_{i+1}-\mathbf{r}_i$. We then assume that within the boundaries of the unit cell the hydrodynamic velocity vector remains contant:

$$\frac{\partial \mathbf{V}_i}{\partial \mathbf{r}_i} \equiv 0 \tag{1}$$

and changes abruptly at the boundary. This discontinuous change of the velocity is illustrated in Fig. 2. If we introduce the center-of-gravity coordinates $\mathbf{r}_{0i}$ of the unit cells, we can then calculate the derivative of this discontinuous function by the finite-difference formula

$$\frac{\partial \mathbf{V}_i}{\partial \mathbf{r}_{0i}} = \frac{\mathbf{V}_i - \mathbf{V}_{i-1}}{\mathbf{r}_{0i} - \mathbf{r}_{0i-1}}, \tag{2}$$

where the sign of the derivatives at the i-th point, calculated according to Eq. (2), does not determine the behavior of the function at the (i + 1)th point, because the behavior of the discontinuity is of a stochastic nature.

The given discontinuous function can be approximated by a smooth curve (Fig. 2), which meets the straight-line segments at the points $\mathbf{r}_{0i}$ in such a way as to satisfy condition (1) [4].

As mentioned above, the sign of the derivative calculated according to Eq. (2) does not determine the behavior of the function in the next cell, hence we determine the magnitude and direction of the function at the point $\mathbf{r}_{0i}$ from the relation

$$\mathbf{V}_i = \mathbf{V}_{i-1} \pm (\mathbf{r}_{0i} - \mathbf{r}_{0i-1}) \frac{\partial \mathbf{V}_{i-1}}{\partial \mathbf{r}_{0i-1}}. \tag{3}$$

If we introduce the variable $\alpha_i$, which is equal to the ratio of the sides of two adjacent cells:

$$\alpha_i = \frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\mathbf{r}_i - \mathbf{r}_{i-1}}, \tag{4}$$

we can then rewrite Eq. (3) as follows:

$$\mathbf{V}_i = \mathbf{V}_{i-1} \pm (1 + \alpha_i)(\mathbf{r}_{0i-1} - \mathbf{r}_{i-1}) \frac{\partial \mathbf{V}_{i-1}}{\partial \mathbf{r}_{0i-1}}. \tag{5}$$

A change of the velocity from $\mathbf{V}_{i-1}$ to $\mathbf{V}_i$ induces an additional rotation of all points of the unit cell about the center of gravity of the adjacent cell with an angular velocity $\omega_{i-1} = \pm \operatorname{rot} \mathbf{V}_{i-1}$. It is obvious that the average linear velocity of all points of the unit cell due to this additional rotation is equal to

$$\mathbf{V}_i^1 = \pm \operatorname{rot} \mathbf{V}_{i-1} \times (\mathbf{r}_{0i} - \mathbf{r}_{0i-1}) = \pm (1 + \alpha_i) \operatorname{rot} \mathbf{V}_{i-1} \times (\mathbf{r}_{0i-1} - \mathbf{r}_{i-1}). \tag{6}$$

Next we must take into account the rate of volume expansion associated with deformation of the unit cell $(\mathbf{r}_{0i}-\mathbf{r}_{0i-1})$:

$$\mathbf{V}_i^2 = \pm (1 + \alpha_i)(\mathbf{r}_{0i-1} - \mathbf{r}_{i-1}) \operatorname{div} \mathbf{V}_{i-1}. \tag{7}$$

Now the total velocity of the i-th unit cell is equal to

$$\mathbf{V} = \mathbf{V}_i + \mathbf{V}_i^1 + \mathbf{V}_i^2. \tag{8}$$

If we use m to denote the mass of one unit cell, we have for its kinetic energy

$$T = \frac{mV^2}{2}$$

$$\approx m\left\{\frac{V_{i-1}^2}{2} + \beta(\mathbf{r}_{0i-1} - \mathbf{r}_{i-1})\left[\mathbf{V}_{i-1}\frac{\partial \mathbf{V}_{i-1}}{\partial \mathbf{r}_{0i-1}} + \operatorname{rot}\mathbf{V}_{i-1}\times\mathbf{V}_{i-1} + \mathbf{V}_{i-1}\operatorname{div}\mathbf{V}_{i-1}\right]\right\}, \tag{9}$$

in which we have introduced the auxiliary symbol

$$\beta = \pm(1+\alpha_i) = \pm\frac{\mathbf{r}_{i+1} - \mathbf{r}_{i-1}}{\mathbf{r}_i - \mathbf{r}_{i-1}}. \tag{10}$$

Let the net-point unit cell be situated in a homogeneous field of surface forces; then its potential energy is determined by the well known formula

$$P = -\mathbf{F}\mathbf{r}_{0i}. \tag{11}$$

Normally the surface forces are specified in terms of their probability density function over the surface or by the stresses:

$$\mathbf{p} = \lim\frac{\Delta \mathbf{F}}{\Delta\sigma} \quad \text{as} \quad \Delta\sigma \to 0, \tag{12}$$

then the following inverse relations hold for the projections of the vector $\mathbf{F}$:

$$\mathbf{F}_x = \mathbf{p}_x \Delta y \Delta z = \frac{m}{\rho}\frac{\partial \mathbf{p}_x}{\partial x},$$

$$\mathbf{F}_y = \frac{m}{\rho}\frac{\partial \mathbf{p}_y}{\partial y}, \quad \mathbf{F}_z = \frac{m}{\rho}\frac{\partial \mathbf{p}_z}{\partial z}. \tag{13}$$

Here $\rho = m/\Delta x \Delta y \Delta z$ is the density of the unit cell. Now, taking (13) into account, we rewrite Eq. (11) as follows:

$$P = -\frac{m}{\rho}\left(\frac{\partial \mathbf{p}_x}{\partial x} + \frac{\partial \mathbf{p}_y}{\partial y} + \frac{\partial \mathbf{p}_z}{\partial z}\right)\mathbf{r}_{0i}. \tag{14}$$

If we adopt $\mathbf{r}_{0i-1}$ and $\mathbf{V}_{i-1}$ as generalized coordinates, where

$$\mathbf{V}_{i-1} = \frac{d\mathbf{r}_{0i-1}}{dt}, \tag{15}$$

the Helmholtz equation for the unit cell assumes the form

$$\frac{\partial H}{\partial \mathbf{r}_{0i-1}} - \frac{d}{dt}\frac{\partial H}{\partial \mathbf{V}_{i-1}} = 0. \tag{16}$$

Here H = P−T is the kinetic potential.

We write down the following requisite equations:

$$\frac{\partial H}{\partial \mathbf{r}_{0i-1}} = -m\left\{\frac{1}{\rho}\left(\frac{\partial \mathbf{p}_x}{\partial x} + \frac{\partial \mathbf{p}_y}{\partial y} + \frac{\partial \mathbf{p}_z}{\partial z}\right)\right.$$
$$\left. + \beta\left[\mathbf{V}_{i-1}\frac{\partial \mathbf{V}_{i-1}}{\partial \mathbf{r}_{0i-1}} + (\operatorname{rot}\mathbf{V}_{i-1}\times\mathbf{V}_{i-1}) + \mathbf{V}_{i-1}\operatorname{div}\mathbf{V}_{i-1}\right]\right\},$$
$$\frac{d}{dt}\frac{\partial H}{\partial \mathbf{V}_{i-1}} = m\frac{d\mathbf{V}_{i-1}}{dt} = m\left[\frac{\partial \mathbf{V}_{i-1}}{\partial t} + (\mathbf{V}_{i-1}\nabla)\mathbf{V}_{i-1}\right],$$
$$\nabla_x = \frac{\partial}{\partial x}, \quad \nabla_y = \frac{\partial}{\partial y}, \quad \nabla_z = \frac{\partial}{\partial z}. \tag{17}$$

Now Eq. (16) may be rewritten in the form

$$\frac{\partial \mathbf{V}_{i-1}}{\partial t} + (1-\beta)(\mathbf{V}_{i-1}\nabla)\mathbf{V}_{i-1} - \beta \mathbf{V}_{i-1} \operatorname{div} \mathbf{V}_{i-1} = \frac{1}{\rho}\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z}\right). \tag{18}$$

Inasmuch as Eq. (3) holds for any two adjacent cells, we can drop the subscripts (i−1) and, invoking the Stokes hypothesis, deduce the following:

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho[(1-\beta)(\mathbf{V}\nabla)\mathbf{V} - \beta \mathbf{V} \operatorname{div} \mathbf{V}] = -\operatorname{grad} p + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial \mathbf{V}}{\partial x} + \operatorname{grad} u\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial \mathbf{V}}{\partial y} + \operatorname{grad} v\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial \mathbf{V}}{\partial z} + \operatorname{grad} w\right)\right] + \operatorname{grad}(\lambda \operatorname{div} \mathbf{V}). \tag{19}$$

Equation (19) was first obtained by Predvoditelev in the above-cited paper [2]. In order to arrive at Kasterin's equation it is required in (19) to set the product V div V equal to zero. This implies that if $\mathbf{V} \neq 0$, at a point of discontinuity the fluid unit cell is replaced by a unit solid.

Note that the equation of continuity has the usual form, since the law of mass conservation holds at the point of discontinuity.

## NOTATION

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| r | radius vector with projections x, y, and z; |
| x, y, z | Cartesian coordinates; |
| V | velocity vector with projections u, v, and w; |
| $r_0$ | radius vector of the center of gravity of the fluid net-point unit cell; |
| T | kinetic energy; |
| P | potential energy; |
| F | surface-force vector. |

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